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SUMMARY OF RESULTS

1. THEORETICAL FORMULAE HAVE BEEN DERIVED SHOWING THE ELASTIC, PLASTIC BEHAVIOR UNDER UNIFORM PRESSURE OF CIRCULAR CYLINDERS COMPOSED OF THREE TYPES OF MATERIAL. THE FIRST IS A MATERIAL CHARACTERIZED BY AN ARBITRARY ISOTROPIC STRESS-STRAIN LAW. IN THIS CASE FORMULAE ARE DERIVED FOR THE BEHAVIOR OF A CYLINDER UNDER THE RELATIVELY COMPLEX TYPE OF DEFORMATION HERE CONSIDERED FROM THE SIMPLE STRESS-STRAIN CURVE OF THE MATERIAL IN TENSION. THE SECOND AND THIRD TYPES OF MATERIAL CONSIDERED ARE PARTICULAR CASES OF THE ABOVE; NAMELY, A MATERIAL WITH NO STRAIN-HARDENING, AND A MATERIAL WITH CONSTANT STRAIN HARDENING.

FOR EACH OF THESE THREE MATERIALS ALL STRESSES AND STRAINS AT A POINT MAY BE COMPUTED IN TERMS OF e_{tb}, THE TANGENTIAL BORE STRAIN, e_z, THE AXIAL STRAIN, p_b, THE EXTERNAL APPLIED PRESSURE, W, THE RATIO OF INITIAL OUTSIDE DIAMETER TO INITIAL INSIDE DIAMETER, AND THE RADIAL DISTANCE OF THE POINT DIVIDED BY THE BORE RADIUS. IN PARTICULAR, THE PRESSURE FACTOR, I.E., THE RATIO OF THE PRESSURE DIFFERENCE TO THE YIELD STRESS, THE OVERALL AXIAL FORCE, AND THE FLOW FACTOR, I.E., THE RATIO IN INCREASE OF INSIDE DIAMETER TO INCREASE OF OUTSIDE DIAMETER, MAY BE FOUND FROM FORMULAS (17), (19a), (22), (25), (26), (30), AND (32).

IN ORDER TO OBTAIN THE CASE OF A CYLINDER YIELDING ONLY PARTIALLY THE CORRESPONDING SOLUTION FOR THE ELASTIC DEFORMATION OF A CIRCULAR CYLINDER IS CONSIDERED, AND A METHOD IS PRESENTED FOR MATCHING THIS ELASTIC SOLUTION TO ANY OF THE PRECEDING PLASTIC ONES. ALL STRESSES AND STRAINS IN BOTH THE ELASTIC AND PLASTIC PORTIONS OF THE PARTIALLY YIELDED CYLINDER MAY BE READILY FOUND FROM THE FORMULAE PRESENTED.

2. GRAPHICAL APPLICATIONS OF THESE FORMULAE WILL BE PRESENTED IN SUBSEQUENT REPORTS.

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TABLE OF CONTENTS

INTRODUCTION

PREVIOUS ORDNANCE WORK CONCERNED WITH THE MATHEMATICAL THEORY OF THE PLASTIC FLOW OF HOLLOW CYLINDERS UNDER UNIFORM INTERNAL AND EXTERNAL PRESSURES HAS BEEN GENERALLY BASED UPON THE ASSUMPTION THAT THE CYLINDER MAINTAINS A FIXED LENGTH DURING THE DEFORMATION. COMPARISON OF THE RESULTS OF THIS SIMPLIFIED THEORY WITH EXPERIMENT SHOWS A SIGNIFICANT DISCREPANCY IN MANY PRACTICAL CASES OF IMPORTANCE. THE PRESENT INVESTIGATION HAS BEEN UNDERTAKEN WITH THE OBJECT OF DERIVING FORMULAE FOR THE STRESSES AND DEFORMATIONS ALLOWING FOR CHANGES IN CYLINDER LENGTH, RETAINING ONLY THE ASSUMPTION THAT THE LONGITUDINAL EXPANSION OR CONTRACTION REMAINS THE SAME THROUGHOUT THE CYLINDER. THESE FORMULAE, EXPANDED AND EXPRESSED IN GRAPHICAL FORM IN LATER REPORTS, WILL PROVIDE IMPROVED CALCULATION METHODS FOR:

1. DESIGN INFORMATION FOR COLDWORKING OF GUN TUBES; THE EFFECT OF A CHANGE IN THE RATIO OF YIELD TO TENSILE STRENGTH UPON PRESENT DESIGN DATA; RESIDUAL STRESSES AFTER COLDWORK;
2. THE ALLOWABLE DEFORMATIONS AND STRENGTH OF GUNS AND SHELLS IN FIRING; THE MECHANISM OF BURSTING;
3. FUTURE SPECIALIZED PROBLEMS, SUCH AS LOCALIZED INTERNAL AND EXTERNAL LOADS.

THE DISCUSSION IN THIS REPORT IS DEVOTED TO A MATHEMATICAL ANALYSIS OF:

1. GENERAL PLASTIC STRESS-STRAIN RELATIONS
2. THE COMPLETE YIELDING OF CIRCULAR CYLINDERS
3. THE PARTIAL YIELDING OF CIRCULAR CYLINDERS

DISCUSSION

1. GENERAL PLASTIC STRESS-STRAIN RELATIONS:

THE FOLLOWING ASSUMPTIONS WILL BE MADE:

a. THE MATERIAL IS CONSIDERED TO FOLLOW THE VON MISES CONDITION AS EXPRESSED IN EQUATIONS (3) AND (4), AND THE CONDITION OF INCOMPRESSIBILITY, WHILE UNDER LOADING BEYOND THE YIELD POINT. BELOW THE YIELD POINT, AND IN RECOVERY DURING UNLOADING, THE MATERIAL WILL BE ASSUMED TO FOLLOW HOOKE'S LAW WITH COMPRESSIBILITY CONSIDERED.

b. ALL QUANTITIES ARE SYMMETRICAL ABOUT THE LONGITUDINAL AXIS OF THE CYLINDER

c. ALL QUANTITIES ARE INDEPENDENT OF POSITION ALONG THE LENGTH OF THE CYLINDER.

d. THE LONGITUDINAL STRAIN, e_z, IS A CONSTANT THROUGHOUT THE WHOLE OF THE CYLINDER; AND ALL PLANE CROSS SECTIONS REMAIN PLANE DURING THE DEFORMATION.

e. THE APPLIED INTERNAL AND EXTERNAL PRESSURES ARE ASSUMED UNIFORM, AND THE LONGITUDINAL FORCES APPLIED AT THE ENDS OF THE CYLINDER ARE ASSUMED TO BE SO DISTRIBUTED THAT ASSUMPTIONS b, c, d, ABOVE, ARE VALID.

LET r REPRESENT THE RADIAL DISTANCE OF A GIVEN POINT FROM THE CYLINDRICAL AXIS, AND LET S_t, S_r, S_z, e_t, e_r, AND e_z BE THE TANGENTIAL, RADIAL, AND AXIAL STRESSES AND STRAINS RESPECTIVELY. CYLINDRICAL COORDINATE PLANES ARE THE PRINCIPAL PLANES OF STRESS AND STRAIN, SO THAT ALL SHEARING STRESSES AND STRAINS ACROSS THESE PLANES VANISH. BY THE ASSUMPTIONS ABOVE, ALL QUANTITIES ARE FUNCTIONS OF r ALONE, AND OF THE USUAL THREE EQUILIBRIUM EQUATIONS FOR STRESSES ONLY ONE REMAINS TO BE SATISFIED:

$$\frac{dS_r}{dr} = \frac{S_t - S_r}{r} \tag{1}$$

FOR PLASTIC DEFORMATION UNDER LOADING, THE EQUATION OF INCOMPRESSIBILITY MUST BE SATISFIED:

$$e_t + e_r + e_z = 0 \tag{2}$$

IN A SIMPLE TENSION TEST THERE IS A DEFINITE FUNCTIONAL RELATION BETWEEN THE STRAIN AND THE CORRESPONDING STRESS. IN A COMPLEX STATE OF STRESS AND STRAIN SUCH AS IS PRESENT IN A YIELDING CYLINDER A SIMILAR RELATION BETWEEN TWO ANALOGOUS QUANTITIES OBTAINED FROM THE COMBINED SYSTEM OF STRESSES AND STRAINS MAY BE WRITTEN.[1] LET

$$S = \frac{1}{\sqrt{2}}\left[(S_t - S_r)^2 + (S_r - S_z)^2 + (S_z - S_t)^2\right]^{1/2} \tag{3a}$$

AND

$$e = \frac{\sqrt{2}}{3}\left[(e_t - e_r)^2 + (e_r - e_z)^2 + (e_z - e_t)^2\right]^{1/2} \tag{3b}$$

WHERE THE POSITIVE SIGNS OF THE RADICALS ARE TO BE TAKEN. THE FUNCTIONAL RELATIONSHIP BETWEEN THESE TWO QUANTITIES MAY BE EXPRESSED BY

$$S = S(e) \tag{3}$$

THE CONSTANT FACTORS IN (3a) AND (3b) HAVE BEEN SO CHOSEN THAT S IS THE SAME FUNCTION OF e THAT STRESS IS OF STRAIN IN A SIMPLE TENSION TEST OF THE MATERIAL. THUS IT IS POSSIBLE IN LATER FORMULAE TO SUBSTITUTE TENSION TEST DATA DIRECTLY.

TWO EQUATIONS MAY BE WRITTEN RELATING THE INDIVIDUAL STRESS AND STRAIN COMPONENTS (ONLY ONE OF WHICH IS AN INDEPENDENT CONDITION, HOWEVER)

$$\frac{S_t - S_r}{e_t - e_r} = \frac{S_r - S_z}{e_r - e_z} = \frac{S_z - S_t}{e_z - e_t} \tag{4}$$

THE RATIO INTRODUCED IN (4) WILL IN GENERAL VARY FROM POINT TO POINT OF THE MATERIAL; HOWEVER, IT MUST BE CONSIDERED AS INTRINSICALLY POSITIVE IF THE MATERIAL IS TO DEFORM IN THE DIRECTION OF LOADING.

FROM (3a), (3b), AND (4) IT FOLLOWS THAT

$$S^2 = (S_t - S_r)^2\left[1 + \left(\frac{S_r - S_z}{S_t - S_r}\right)^2 + \left(\frac{S_z - S_t}{S_t - S_r}\right)^2\right]$$

$$= \left(\frac{S_t - S_r}{e_t - e_r}\right)^2 (e_t - e_r)^2\left[1 + \left(\frac{e_r - e_z}{e_t - e_r}\right)^2 + \left(\frac{e_z - e_t}{e_t - e_r}\right)^2\right]$$

$$= \left(\frac{S_t - S_r}{e_t - e_r}\right)^2 \frac{9}{2} e^2 \tag{5}$$

WHENCE $\frac{S_t - S_r}{e_t - e_r} = \frac{2}{3}\,\frac{S(e)}{e}$ (5a)

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[1] A. NADAI, ON THE MECHANICS OF THE PLASTIC STATE OF METALS, ASME TRANSACTIONS, 1930 V. 52 (1)

2. THE COMPLETE YIELDING OF CIRCULAR CYLINDERS

IF AN INTERNAL PRESSURE IS GRADUALLY APPLIED TO A CYLINDER, STARTING FROM AN UNSTRESSED STATE, THEN AT FIRST THE WHOLE CYLINDER WILL BE DEFORMED ELASTICALLY. AS THE PRESSURE INCREASES THE MORE HIGHLY STRESSED INNER PORTION WILL YIELD PLASTICALLY, WHILE THE OUTSIDE STILL REMAINS ELASTIC. THE TWO REGIONS WILL BE SEPARATED BY A CYLINDRICAL BOUNDARY, FORMING, IN EFFECT, TWO DISTINCT TUBES. WHEN STILL MORE PRESSURE IS APPLIED, THE INNER REGION WILL INCREASE UNTIL FINALLY THE WHOLE TUBE IS IN THE PLASTIC STATE. THE INTERMEDIATE STATE IS TERMED PARTIAL YIELDING, IN CONTRAST TO THE FINAL STATE OF COMPLETE YIELDING.

THE CASE OF COMPLETE YIELDING WILL FIRST BE TREATED IN WHICH ONLY THE APPROPRIATE SOLUTIONS OF THE GENERAL PLASTIC STRESS–STRAIN RELATIONS NEED BE FOUND. IN PARTIAL YIELDING, HOWEVER, THE SOLUTION FOR THE ELASTIC CYLINDER WILL BE OBTAINED SEPARATELY, WHEREUPON IT MAY BE MATCHED WITH THE EARLIER PLASTIC SOLUTION SO AS TO MEET THE CONDITIONS OF THE PROBLEM.

A. GENERAL FORMAL SOLUTION FOR MATERIAL WITH AN ARBITRARY ISOTROPIC[2] STRESS–STRAIN LAW:

LET $U(r)$ BE THE RADIAL DISPLACEMENT OF A POINT DURING THE DEFORMATION. THEN

$$e_t = \frac{U}{r}, \quad e_r = \frac{dU}{dr}, \quad e_z = \text{AN ARBITRARY CONSTANT} \tag{6}$$

THE INCOMPRESSIBILITY RELATION THEN BECOMES

$$\frac{dU}{dr} + \frac{U}{r} + e_z = 0 \tag{7}$$

SOLVING (7) FOR $U(r)$, AND LETTING e_{ta} = THE TANGENTIAL STRAIN AT THE BORE RADIUS ($r = a$), $R = \frac{r}{a}$, AND $\bar{e}_z = \frac{e_z}{e_{ta}}$, THE STRAINS MAY BE WRITTEN

$$\left.\begin{aligned} e_t &= \frac{e_{ta}}{2}\left[- \bar{e}_z + \frac{2 + \bar{e}_z}{R^2}\right] \\ e_r &= \frac{e_{ta}}{2}\left[- \bar{e}_z - \frac{2 + \bar{e}_z}{R^2}\right] \\ e_z &= e_{ta}\,\bar{e}_z \end{aligned}\right\} \tag{8}$$

---

[2] BY ISOTROPY IT IS MEANT ONLY THAT THE INITIAL PROPERTIES OF THE MATERIAL ARE INDEPENDENT OF DIRECTION.

THE STRAIN DIFFERENCES MAY THEN BE WRITTEN

$$\left.\begin{aligned} e_t - e_r &= \frac{e_{ta}}{R^2}(2 + \bar{e}_z) \\ e_r - e_z &= \frac{e_{ta}}{2}\left[- 3\bar{e}_z - \frac{2 + \bar{e}_z}{R^2}\right] \\ e_z - e_t &= \frac{e_{ta}}{2}\left[+ 3\bar{e}_z - \frac{2 + \bar{e}_z}{R^2}\right] \end{aligned}\right\} \qquad (9)$$

FROM WHICH e MAY BE IMMEDIATELY COMPUTED IN TERMS OF R.

$$e = |e_{ta}| \left\{ \frac{(2 + \bar{e}_z)^2}{3R^4} + \bar{e}_z^{\,2} \right\}^{1/2} \qquad (10)$$

WRITING $\bar{e} = \frac{e}{|e_{ta}|}$, (10) BECOMES

$$\bar{e} = \left\{ \frac{(2 + \bar{e}_z)^2}{3R^4} + \bar{e}_z^{\,2} \right\}^{1/2} \qquad (10a)$$

FROM (9) IT IS EVIDENT THAT THE SIGN OF $e_t - e_r$, HENCE OF $S_t - S_r$, AND HENCE OF $\frac{dS_r}{dr}$, IS THE SAME THROUGHOUT THE CYLINDER, AND DEPENDS ONLY UPON THE SIGNS OF e_{ta} AND OF $2 + \bar{e}_z$. IF THE CYLINDER IS UNDER LONGITUDINAL TENSION, WITH NO INTERNAL AND EXTERNAL PRESSURES, $\bar{e}_z = -2$, OR $e_{ta} = -\frac{1}{2}e_z$, AS WOULD BE EXPECTED FOR A MATERIAL WITH POISSON'S RATIO OF 0.5. FOR THE CASE OF PRIMARY INTEREST, BORE EXPANSION UNDER INTERNAL PRESSURE, e_{ta} WILL BE POSITIVE, AND $\bar{e}_z$ WILL BE GREATER THAN THIS VALUE OF -2. IT WILL BE ASSUMED, THEREFORE, THAT BOTH e_{ta} AND $2 + \bar{e}_z$ ARE POSITIVE THROUGHOUT THE REST OF THIS REPORT. THE DEVELOPMENT IS IN NO SENSE LIMITED BY THIS ASSUMPTION, BUT MAY BE APPLIED TO THE MOST GENERAL CASE (THAT IS, FOR ANY COMBINATION OF INTERNAL OR EXTERNAL PRESSURES OR TENSIONS) WITH ONLY MINOR CHANGES IN SIGNS.

THE STRAINS (HENCE e, AND S(e)) HAVE NOW BEEN EXPRESSED AS FUNCTIONS OF R. THUS FROM (1) AND (5a) IT IS EVIDENT THAT S_r AND THE REMAINING STRESSES MAY READILY BE FOUND IN TERMS OF R. RATHER THAN DO THIS DIRECTLY, IT WILL BE MORE CONVENIENT TO EMPLOY $\bar{e}$ AS THE INDEPENDENT VARIABLE. R WILL THEREFORE BE EXPRESSED AS A FUNCTION OF $\bar{e}$ FROM (10a). THUS

$$R^2 = \frac{2 + \bar{e}_z}{\sqrt{3}\left(\bar{e}^2 - \bar{e}_z^{\,2}\right)^{1/2}} \qquad (11)$$

AND ALSO $\dfrac{dr}{r} = \dfrac{dR^2}{2R^2} = -\dfrac{1}{2}\,\dfrac{\bar{e}\,d\bar{e}}{\bar{e}^2 - \bar{e}_z^2}$

AND $\quad R\,dR = \dfrac{dR^2}{2} = -\dfrac{(2 + \bar{e}_z)\bar{e}\,d\bar{e}}{2\sqrt{3}\left[\bar{e}^2 - \bar{e}_z^2\right]^{3/2}}$.

FROM (9), THEREFORE,

$$
\left.
\begin{aligned}
e_t - e_r &= e_{ta}\sqrt{3}(\bar{e}^2 - \bar{e}_z^2)^{1/2} \\
e_r - e_z &= \frac{e_{ta}}{2}\left\{ -3\bar{e}_z - \sqrt{3}\left(\bar{e}^2 - \bar{e}_z^2\right)^{1/2} \right\} \\
e_z - e_t &= \frac{e_{ta}}{2}\left\{ +3\bar{e}_z - \sqrt{3}\left(\bar{e}^2 - \bar{e}_z^2\right)^{1/2} \right\}
\end{aligned}
\right\} \qquad (13)
$$

BY (5a) AND (13),

$$
\left.
\begin{aligned}
S_t - S_r &= \frac{2S(e)}{\sqrt{3}\bar{e}}\left(\bar{e}^2 - \bar{e}_z^2\right)^{1/2} \\
S_r - S_z &= \frac{S(e)}{3\bar{e}}\left\{ -3\bar{e}_z - \sqrt{3}\left(\bar{e}^2 - \bar{e}_z^2\right)^{1/2} \right\} \\
S_z - S_t &= \frac{S(e)}{3\bar{e}}\left\{ +3\bar{e}_z - \sqrt{3}\left(\bar{e}^2 - \bar{e}_z^2\right)^{1/2} \right\}
\end{aligned}
\right\} \qquad (14)
$$

WITH THESE PRELIMINARY RELATIONS S_r MAY NOW BE FOUND. SUBSTITUTION FROM (14) INTO (1) AND INTEGRATION FROM THE OUTSIDE WALL OF THE CYLINDER ($R = W$) TO A POINT P, YIELDS BY (12),

$$
S_r \Big]_{-P_b}^{S_r} = \int_W^R \frac{S_t - S_r}{R}\, dR
$$

$$
S_r = -P_b + \frac{1}{\sqrt{3}}\int_{\bar{e}}^{\bar{e}_b} \frac{S(e_{ta}\bar{e})}{\left[\bar{e}^2 - \bar{e}_z^2\right]^{1/2}}\, d\bar{e} \qquad (15)
$$

WHERE P_b IS THE EXTERNAL PRESSURE ON THE CYLINDER, $\bar{e}_b$ IS THE VALUE OF $\bar{e}$ AT THE OUTSIDE WALL, GIVEN FROM (10a) BY

$$
\bar{e}_b = \left[\frac{(2 + \bar{e}_z)^2}{3W^4} + \bar{e}_z^2 \right]^{1/2} \qquad (16)
$$

WHILE $\bar{e}$ IS GIVEN IN TERMS OF R IN (10a). S IS STILL A FUNCTION OF e ($= e_{ta}\bar{e}$) IN (15). THE SCALE MUST THEREFORE BE CONVERTED, IF THE INTEGRATION IS TO BE PERFORMED NUMERICALLY OR GRAPHICALLY. IN USING THE "BARRED" STRAIN EXPRESSIONS, RELATIVE STRAINS HAVE BEEN INTRODUCED, THAT IS, STRAINS PER UNIT OF BORE EXPANSION. THE FUNCTION S, HOWEVER, DEPENDS ON THE TOTAL EXTENT OF STRAIN AT A GIVEN POINT, RATHER THAN UPON SUCH A RELATIVE STRAIN.

THE INTERIOR PRESSURE ON THE CYLINDER MAY BE COMPUTED READILY FROM (15). LETTING P_a BE THE INTERIOR PRESSURE, AND $P^* = \frac{P_a - P_b}{S_y}$, WHERE S_y IS THE YIELD STRESS OF THE MATERIAL, IT FOLLOWS THAT

$$P^* = \frac{1}{\sqrt{3}S_y} \int_{\bar{e}_b}^{\bar{e}_a} \frac{S(e_{ta}\bar{e})}{\left[\bar{e}^2 - \bar{e}_z^2\right]^{1/2}} d\bar{e} \tag{17}$$

TO COMPUTE THE TOTAL LONGITUDINAL FORCE OPERATING OVER THE ENDS OF THE CYLINDER, S_z MUST BE INTEGRATED OVER THE CYLINDRICAL CROSS SECTION. LET F BE THE TOTAL ENDWISE FORCE, AND SET $F^* = \frac{F}{S_y \pi a^2}$

THEN

$$F^* = \frac{2}{S_y} \int_1^W S_z R dR \tag{18}$$

SUBSTITUTION FROM (12) AND (14) GIVES, THEREFORE,

$$F^* = \frac{1}{S_y} \int_{\bar{e}_b}^{\bar{e}_a} \left\{ S_r + \frac{S(e)}{3\bar{e}} \left[3\bar{e}_z + \sqrt{3}(\bar{e}^2 - \bar{e}_z^2)^{1/2} \right] \right\} \frac{(2 + \bar{e}_z)\bar{e} d\bar{e}}{\sqrt{3}\left(\bar{e}^2 - \bar{e}_z^2\right)^{3/2}} \tag{19}$$

THIS EXPRESSION MAY BE SIMPLIFIED BY INTEGRATING THE TERM IN S_r BY PARTS. USING THE ABOVE DEFINITION FOR P^* AND THE RELATIONS (FROM (11))

$$1 = \frac{2 + \bar{e}_z}{\sqrt{3}\left(\bar{e}_a^2 - \bar{e}_z^2\right)^{1/2}} \quad \text{AND} \quad W^2 = \frac{2 + \bar{e}_z}{\sqrt{3}\left(\bar{e}_b^2 - \bar{e}_z^2\right)^{1/2}}$$

TO SIMPLIFY THE INTEGRATED TERMS, THE FINAL EXPRESSION MAY BE OBTAINED,

$$F^* = P^* - \frac{P_b}{S_y}(W^2 - 1) + \frac{\bar{e}_z(2 + \bar{e}_z)}{\sqrt{3}S_y} \int_{\bar{e}_b}^{\bar{e}_a} \frac{S(e_{ta}\bar{e})}{\left(\bar{e}^2 - \bar{e}_z^2\right)^{3/2}} d\bar{e} \tag{19a}$$

ONE MORE QUANTITY OF INTEREST REMAINS. THIS IS THE SO-CALLED FLOW FACTOR (FF); THAT IS, THE RATIO OF THE INCREASE OF THE INSIDE DIAMETER TO THE INCREASE OF THE OUTSIDE DIAMETER. THIS MAY EVIDENTLY BE WRITTEN

$$FF = \frac{e_{ta}}{W e_{tb}} \tag{20}$$

THROUGH CONSIDERATION OF (6), WHERE e_{tb} IS THE VALUE OF e_t AT $\underline{R} = W$. EMPLOYING (8),

$$e_{tb} = \frac{e_{ta}}{2} \left\{ -\bar{e}_z + \frac{2 + \bar{e}_z}{W^2} \right\} \tag{21}$$

WHENCE $FF = \frac{2W}{2 - \bar{e}_z(W^2 - 1)}$ (22)

EXAMINATION OF THE FORMULAE THUS FAR OBTAINED SHOWS THAT IF e_{za}, e_{ta}, AND P_b ARE SPECIFIED, THEN ALL OTHER STRESSES AND STRAINS MAY READILY BE DETERMINED. THIS FACT WILL BE OF IMPORTANCE IN THE CASE OF PARTIAL YIELDING TO BE DISCUSSED LATER.

B. SOLUTION FOR MATERIAL YIELDING UNDER CONSTANT STRESS:

IN THIS CASE, THE FUNCTION $S(e)$ ASSUMES ITS SIMPLEST FORM, NAMELY

$$S(e) = \text{A CONSTANT YIELD STRESS, } S_y. \tag{23}$$

THE FORMULAE FOR S_r, F^*, AND P^* MAY THEREFORE BE EVALUATED ANALYTICALLY FROM THE INTEGRAL FORMS (15), (17), AND (19a). THUS, FOR S_r IS OBTAINED AN EXPRESSION[3] EQUIVALENT TO ONE GIVEN BY NADAI AND LODE[4]:

$$S_r = -P_b + \frac{S_y}{\sqrt{3}}\left[\cosh^{-1}\left(\frac{\bar{e}_b}{\bar{e}_z}\right) - \cosh^{-1}\left(\frac{\bar{e}}{\bar{e}_z}\right)\right] \tag{24}$$

FOR P^*

$$P^* = \frac{1}{\sqrt{3}}\left[\cosh^{-1}\left(\frac{\bar{e}_a}{\bar{e}_z}\right) - \cosh^{-1}\left(\frac{\bar{e}_b}{\bar{e}_z}\right)\right] \tag{25}$$

AND FOR F^*

$$F^* = -\frac{P_b(W^2 - 1)}{S_y} + P^* + \frac{W^2\bar{e}_b - \bar{e}_a}{\bar{e}_z} \tag{26}$$

SINCE THE FLOW FACTOR DOES NOT DEPEND UPON THE PARTICULAR FORM ASSUMED FOR THE STRESS-STRAIN LAW OF THE MATERIAL, BUT ONLY UPON THE INCOMPRESSIBILITY AS EXPRESSED THROUGH THE STRAINS, THIS QUANTITY WILL HAVE THE SAME VALUE (22) AS BEFORE.

FOR A GIVEN VALUE OF EXTERNAL PRESSURE, THE EQUATION $F^* = 0$ GIVES A RELATION BETWEEN $\bar{e}_z$ AND W. THIS RELATION IS OF PARTICULAR PRACTICAL IMPORTANCE, AND IT WILL BE SHOWN IN GRAPHICAL FORM IN SUBSEQUENT REPORTS THAT THIS RELATION AGREES BETTER WITH EXPERIMENT THAN THE CUSTOMARY APPROXIMATION ASSUMING $e_z = 0$.

C. SOLUTION FOR CONSTANT STRAIN HARDENING:

IN THIS CASE[5]

$$S(e) = S_y + E_s e \tag{27}$$

WHERE E_s IS A SLOPE ANALOGOUS TO THE ORDINARY YOUNG'S MODULUS.

IT IS EVIDENT THAT THE EXPRESSIONS FOR S_r, P^*, AND F^* WILL CONTAIN THE EXPRESSIONS COMPUTED ABOVE FOR YIELDING UNDER CONSTANT STRESS AS

---

[3] FOR COMPUTATIONAL PURPOSES IT IS OFTEN OF CONVENIENCE TO EMPLOY THE RELATION $\cosh^{-1}\left(\frac{\bar{e}}{\bar{e}_z}\right) = \sinh^{-1}\left(\frac{\bar{e}^2}{\bar{e}_z^2} - 1\right)^{1/2} = \sinh^{-1}$ [illegible]

[4] A. NADAI, PLASTIC BEHAVIOR OF METALS IN THE STRAIN-HARDENING RANGE. PART I. JOURNAL OF APPLIED PHYSICS, MARCH, 1937.

[5] S_y, AS HERE DEFINED, DIFFERS NEGLIGIBLY FROM THE ORDINARY YIELD POINT OF THE MATERIAL.

SPECIAL CASES IN WHICH $E_a = 0$. THUS IT WILL BE NECESSARY TO COMPUTE ONLY THOSE TERMS CONTAINING E_a AS EXPRESSIONS TO BE ADDED TO THE TERMS PREVIOUSLY COMPUTED.

THUS, TO S_r MUST BE ADDED

$$\frac{1}{\sqrt{3}}\int_{\bar{e}_a}^{\bar{e}_b} \frac{e_{ta}E_a\ \bar{e}d\bar{e}}{\left(\bar{e}^2 - \bar{e}_z^{\,2}\right)^{1/2}} = \frac{e_{ta}E_a}{\sqrt{3}}\left\{(\bar{e}_b^{\,2} - \bar{e}_z^{\,2})^{1/2} - (\bar{e}^2 - \bar{e}_z^{\,2})^{1/2}\right\}$$

$$= \frac{e_{ta}E_a(2 + \bar{e}_z)}{3}\left[\frac{1}{W^2} - \frac{1}{R^2}\right] \qquad (28)$$

WHENCE $\quad S_r = -P_b + \frac{S_y}{\sqrt{3}}\left[\cosh^{-1}\left(\frac{\bar{e}_b}{\bar{e}_z}\right) - \cosh^{-1}\left(\frac{\bar{e}}{\bar{e}_z}\right)\right] + \frac{e_{ta}E_a(2 + \bar{e}_z)}{3}\left[\frac{1}{W^2} - \frac{1}{R^2}\right] \quad (29)$

P^+ THEREFORE IS GIVEN BY

$$P^+ = \frac{1}{\sqrt{3}}\left[\cosh^{-1}\left(\frac{\bar{e}_a}{\bar{e}_z}\right) - \cosh^{-1}\left(\frac{\bar{e}_b}{\bar{e}_z}\right)\right] + \frac{e_{ta}E_a(2 + \bar{e}_z)}{3S_y}\left[1 - \frac{1}{W^2}\right] \qquad (30)$$

SIMILARLY THE NECESSARY EXPRESSION TO BE ADDED TO F^+ IS

$$\frac{e_{ta}E_a\ \bar{e}_z}{S_y}\left(W^2 - 1\right) \qquad (31)$$

WHENCE

$$F^+ = \frac{P_b(W^2 - 1)}{S_y} + P^+ + \frac{W^2\bar{e}_b - \bar{e}_a}{\bar{e}_z} + \frac{e_{ta}E_a\bar{e}_z}{S_y}\left(W^2 - 1\right) \qquad (32)$$

WHERE P^+ IS GIVEN BY (30)

D. <u>THIN WALLED CYLINDERS:</u>

IF IT BE PRESUMED THAT WITH A THIN WALLED CYLINDER THE STRAIN DOES NOT VARY MUCH WITH R, THEN THE ABOVE SOLUTION FOR CONSTANT STRAIN HARDENING IS APPLICABLE.

THIN WALLED CYLINDERS MAY BE ANALYZED DIRECTLY FROM FIRST ORDER APPROXIMATIONS TO THE GENERAL PLASTIC STRESS STRAIN RELATIONS HOWEVER, IF IT IS CONSIDERED THAT FOR SUCH TUBES ALL STRESSES MUST VARY LINEARLY ACROSS THE WALL, INDEPENDENTLY OF THE STRESS STRAIN RELATION OF THE MATERIAL.

SUPPOSE, FOR SIMPLICITY, THAT $P_b = 0$.

THEN, OBVIOUSLY,

$$S_r = -P_a\,\frac{W - R}{W - 1} \qquad (33)$$

AND FROM THE EQUILIBRIUM EQUATION $\frac{d(RS_r)}{dR_o} = S_\theta$

$$S_\theta = \frac{P_a}{W - 1}\ (2R - W) \qquad (34)$$

WHICH REDUCES TO THE USUAL BOILER FORMULA AT MIDWALL AND WHICH LIKEWISE

COULD HAVE BEEN WRITTEN DOWN AT ONCE.

NOW S_z SHOULD BE ZERO AT THE CENTER OF THE WALL, $R = \frac{W+1}{2}$, FOR ZERO AXIAL FORCE. ALSO, IN (9), $\frac{1}{R^2} \cong 3 - 2R$ WHICH IS $2 - W$ AT $R = \frac{W+1}{2}$. HENCE THE EXPRESSION

$$\frac{S_z - S_r}{e_z - e_r} = \frac{S_z - S_t}{e_z - e_t}$$

APPLIED TO THE CENTER OF THE WALL BECOMES

$$e_z = \frac{-2 + 3(W-1) - (W-1)^2}{4 + 2(W-1)^2} = -\frac{1}{2} + \frac{3}{4}(W-1) \qquad (35)$$

TO THE FIRST ORDER IN WALL THICKNESS.

3. THE PARTIAL YIELDING OF CIRCULAR CYLINDERS

THE CYLINDER WILL NOW BE CONSIDERED AS COMPOSED OF TWO DISTINCT TUBES, AN INNER PLASTIC ONE AND AN OUTER ELASTIC ONE. IT WILL BE FOUND POSSIBLE TO DETERMINE A SYSTEM OF STRESSES AND STRAINS IN EACH CYLINDER SUCH THAT THE APPROPRIATE EXTERNAL BOUNDARY CONDITIONS ARE SATISFIED, AND SUCH THAT THE CONDITIONS AT THE CONTIGUOUS SURFACE OF THE TWO TUBES ARE COMPATIBLE WITH THOSE OF A SINGLE SOLID BODY. IN THIS SOLUTION THE FOLLOWING ASSUMPTIONS WILL BE MADE:

a THE PRESSURE EXTERNAL TO THE ELASTIC CYLINDER IS ZERO.
b THE ONLY REACTION BETWEEN THE TWO CYLINDERS IS THAT OF A MUTUAL NORMAL PRESSURE, P_b.
c THE LONGITUDINAL EXTENSIONS, e_z, IN THE TWO CYLINDERS ARE THE SAME AND CONSTANT THROUGHOUT.
d THE TANGENTIAL STRAIN AT THE ELASTIC-PLASTIC BOUNDARY, $R = R_b$, IS THE SAME IN BOTH CYLINDERS.
e THE MATERIAL AT THE INNER BOUNDARY OF THE ELASTIC CYLINDER IS JUST AT THE YIELD POINT OF THE MATERIAL; THAT IS, THE ELASTIC STRESSES AT THIS POINT SATISFY EQUATIONS (3).

LET W BE NOW THE OVERALL WALL RATIO OF THE COMPLETE CYLINDER, AND LET R_b BE THE WALL RATIO OF THE PLASTIC PORTION ALONE, THE QUANTITY WHICH WAS FORMERLY DENOTED BY W. IT HAS ALREADY BEEN SHOWN THAT FOR GIVEN VALUES OF e_z, e_{ta}, P_b, AND R_b THE PLASTIC SOLUTION IS COMPLETELY DETERMINED, FOR A MATERIAL WITH A GIVEN TYPE OF STRESS-STRAIN LAW. IF THE SYMBOL e_z IS TAKEN AS REPRESENTING BOTH PLASTIC AND ELASTIC LONGITUDINAL STRAIN, THEN CONDITION c WILL BE SATISFIED. THE SOLUTION IS THEREFORE CONSIDERED FOR AN ELASTIC TUBE UNDER PRESSURES P_b AND ZERO, INTERNALLY AND EXTERNALLY, OF WALL RATIO $\frac{W}{R_b}$, AND WITH THE PRESCRIBED LONGITUDINAL STRAIN, e_z. IT IS READILY SEEN THAT THERE ARE LEFT ONLY CONDITIONS d AND e TO BE SATISFIED, WITH TWO VARIABLES, P_b

AND R_b, STILL TO BE DETERMINED.

THE WELL KNOWN SOLUTION FOR AN ELASTIC TUBE UNDER AN INTERNAL PRESSURE, P_b, UNDER NO EXTERNAL PRESSURE, AND WITH A GIVEN LONGITUDINAL FORCE, F_b, BECOMES, IN THE NOTATION USED HERE,

$$\left.\begin{aligned} S_t &= \frac{P_b R_b^2}{W^2 - R_b^2}\left[1 + \frac{W^2}{R^2}\right] \\ S_r &= \frac{P_b R_b^2}{W^2 - R_b^2}\left[1 - \frac{W^2}{R^2}\right] \\ S_z &= \frac{F_b}{\pi(c^2 - b^2)} = \frac{S_y F^+_b}{W^2 - R_b^2} \end{aligned}\right\} \tag{36}$$

WHERE $F^+_b = \frac{F_b}{S_y \pi a^2}$

LET E BE YOUNG'S MODULUS FOR THE MATERIAL, AND μ BE POISSON'S RATIO. THEN THE STRAINS ARE GIVEN BY

$$\left.\begin{aligned} e_t &= \frac{1}{E}\left\{S_t - \mu(S_r + S_z)\right\} \\ e_r &= \frac{1}{E}\left\{S_r - \mu(S_t + S_z)\right\} \\ e_z &= \frac{1}{E}\left\{S_z - \mu(S_t + S_r)\right\} \end{aligned}\right\} \tag{37}$$

LET e_{tb} BE THE VALUE OF e_t AT $R = R_b$, AND LET

$$\bar{e}_{tb} = \frac{e_{tb}}{e_{ta}} = \frac{1}{2}\left\{-\bar{e}_z + \frac{2 + \bar{e}_z}{R_b^2}\right\} \tag{38}$$

AS GIVEN BY (8) FOR THE PLASTIC REGION. WRITING $P^+_b = \frac{P_b}{S_y}$, IT FOLLOWS, UPON SUBSTITUTION FROM (36) INTO (37), THAT

$$\left(\frac{Ee_{ta}}{S_y}\right)\bar{e}_{tb} = \frac{P^+_b}{W^2 - R_b^2}\left\{(1 - \mu)R_b^2 + (1 + \mu)W^2\right\} - \frac{\mu F^+_b}{W^2 - R_b^2} \tag{39}$$

$$\left(\frac{Ee_{ta}}{S_y}\right)\bar{e}_z = \frac{F^+_b}{W^2 - R_b^2} - \frac{2\mu P^+_b R_b^2}{W^2 - R_b^2} \tag{40}$$

SOLVING (39) AND (40) FOR P^+_b AND F^+_b,

$$P^+_b = \frac{(W^2 - R_b^2)\left(\frac{Ee_{ta}}{S_y}\right)\left[\bar{e}_{tb} + \mu\bar{e}_z\right]}{\left\{(1 - \mu - 2\mu^2)R_b^2 + (1 + \mu)W^2\right\}} \tag{41}$$

$$F^+_b = (W^2 - R_b^2)\left(\frac{Ee_{ta}}{S_y}\right)\bar{e}_z + 2\mu R_b^2 P^+_b \tag{42}$$

NOW CONSIDER W, R_b, AND $\bar{e}_z$ AS THE GIVEN QUANTITIES, TOGETHER WITH THE PHYSICAL PROPERTIES OF THE MATERIAL. THEN FROM (38), (41), AND (42) P^+_b AND F^+_b EACH BECOME A KNOWN QUANTITY MULTIPLIED BY THE FACTOR $\frac{Ee_{ta}}{S_y}$. SUBSTITUTING FROM (36) INTO (3), CONDITION e GIVES

$$\frac{(P^*_b)^2(3W^4 + R_b^4) - 2P^*_b F^*_b R_b^2 + (F^*_b)^2}{(W^2 - R_b^2)^2} = 1 \qquad (43)$$

SINCE $\frac{Ee_{ta}}{S_y}$ IS A FACTOR OF BOTH P^*_b AND F^*_b, AS JUST SHOWN, THE ABOVE EXPRESSION MAY BE SOLVED FOR $\frac{Ee_{ta}}{S_y}$ IN TERMS OF KNOWN QUANTITIES, AND HENCE P^*_b AND F^*_b MAY BE DETERMINED. THIS HAVING BEEN DONE, THE ELASTIC STRESSES AND STRAINS MAY BE FOUND FROM (36) AND (37).

HAVING NOW FOUND THE ELASTIC QUANTITIES, IN TERMS OF W, R_b, AND e_z, IT IS POSSIBLE TO RETURN TO THE CONSIDERATION OF THE PLASTIC PORTION OF THE CYLINDER. P_b, e_z, e_{ta}, AND R_b ARE NOW FIXED, SO THAT THE PLASTIC SOLUTION IS DETERMINED AS BEFORE. THUS THE INITIAL SPECIFICATION OF W, R_b, AND e_z IS SUFFICIENT TO DETERMINE THE SOLUTION COMPLETELY. THE OVERALL LONGITUDINAL FORCE COEFFICIENT MAY BE FOUND AS THE SUM OF F^*_a AND F^*_b, AND THE INTERNAL PRESSURE MAY BE FOUND FROM PLASTIC CONSIDERATIONS (CF. (17) OR (25)).

THIS APPROACH PERMITS THE SIMPLE CONSTRUCTION OF CURVES FROM WHICH THE NECESSARY DATA MAY BE TAKEN DIRECTLY. THUS IF W, e_{ta}, P_a, AND EITHER OF e_z OR F^* ARE PRESCRIBED, R_b, P^*_b, AND ALL DEPENDENT QUANTITIES MAY BE FOUND SUCCESSIVELY. THIS SOLUTION, THEN, PRESENTED IN GRAPHICAL FORM, WILL GIVE ALL THE DATA NEEDED FOR DESIGN PURPOSES.

THE SOLUTIONS HERE PRESENTED, EXPRESSED IN GRAPHICAL FORM, WILL FORM THE SUBJECT MATTER FOR SUBSEQUENT REPORTS. THEIR APPLICATION TO PROBLEMS IN THE DESIGN OF GUNS AND SHELLS WILL BE DISCUSSED IN THOSE REPORTS.

GLOSSARY OF SYMBOLS EMPLOYED

r, θ, z = CYLINDRICAL COORDINATES

a = INTERIOR RADIUS OF CYLINDER, BORE RADIUS

b = EXTERIOR RADIUS OF CYLINDER, OR EXTERNAL RADIUS OF THE PLASTIC DOMAIN IN PARTIAL YIELDING

$U(r)$ = RADIAL DISPLACEMENT OF AN ARBITRARY POINT

$R = \frac{r}{a}$ = RADIAL DISTANCE OF AN ARBITRARY POINT RELATIVE TO THE BORE RADIUS

$R_b = \frac{b}{a}$ = RADIUS OF ELASTIC PLASTIC BOUNDARY CIRCLE IN PARTIAL YIELDING RELATIVE TO BORE RADIUS

W = WALL RATIO, RATIO OF EXTERNAL TO INTERNAL RADIUS

S_t, S_r, S_z = TANGENTIAL (θ), RADIAL (r), AND LONGITUDINAL (z) STRESSES

e_t, e_r, e_z = TANGENTIAL, RADIAL, AND LONGITUDINAL STRAINS

S_y = YIELD STRESS OF THE MATERIAL IN TENSION

e_{ta}, e_{tb} = e_t MEASURED AT $r = a$ AND $r = b$

$$S = \frac{1}{\sqrt{2}}\left[(S_t - S_r)^2 + (S_r - S_z)^2 + (S_z - S_t)^2\right]^{1/2} = A$$

MEASURE OF THE COMBINED STRESS AT A POINT

$$e = \frac{\sqrt{2}}{3}\left[(e_t - e_r)^2 + (e_r - e_z)^2 + (e_z - e_t)^2\right]^{1/2} = A$$

MEASURE OF THE COMBINED STRAIN AT A POINT

$S(e)$ = GENERAL FUNCTIONAL RELATION ASSUMED TO EXIST BETWEEN S AND e

$\bar{e} = \frac{e}{e_{ta}}$

$\bar{e}_z = \frac{e_z}{e_{ta}}$

$\bar{e}_{tb} = \frac{e_{tb}}{e_{ta}}$

$\bar{e}_a, \bar{e}_b$ = $\bar{e}$ MEASURED AT $r = a$ AND $r = b$

P_a = INTERNAL PRESSURE ON THE CYLINDER

P_b = EXTERNAL PRESSURE ON THE CYLINDER, OR PRESSURE AT THE ELASTIC PLASTIC BOUNDARY IN PARTIAL YIELDING

F = OVERALL LONGITUDINAL FORCE ACTING OVER THE ENDS OF THE CYLINDER

F_a = TOTAL LONGITUDINAL FORCE OVER THE PLASTIC REGION ALONE

F_b = TOTAL LONGITUDINAL FORCE OVER THE ELASTIC REGION ALONE

$F^*_a, F^*_b, F^* = \frac{F_a}{S_y \pi a^2}, \frac{F_b}{S_y \pi a^2}, \frac{F}{S_y \pi a^2}$

P^* = PRESSURE FACTOR = $\frac{\text{INTERNAL PRESSURE} - \text{EXTERNAL PRESSURE}}{S_y}$

F = FLOW FACTOR = RATIO OF INTERNAL TO EXTERNAL EXP[illegible]

E, ν = YOUNG'S MODULUS AND POISSON'S RATIO

[illegible] OF STRAIN HARDENING [illegible]